Theorem. A.s., SLE_κ is
1) A simple curve when $\kappa \le 4$, $|\gamma(t)| \to \infty$ as $t \to \infty$.
2) A self-touching curve, $4 < \kappa < 8$ $\bigcup \overline{K_t} = \overline{\mathbb{H}}$.
3) A space-filling curve $\kappa \ge 8$.
$\gamma[0,\infty] = \overline{\mathbb{H}}$.

Proof Consider $h_t(z) := g_{t/\kappa}(z) - B_t$.

Then $dh_t(z) = \frac{2/\kappa}{g_{t/\kappa}(z) - B_t} dt - dB_t = \frac{2/\kappa}{h_t(z)} - dB_t \overset{law}{=} \frac{2/\kappa}{h_t(z)} + dB_t$.

Bessel process with $a = 2/\kappa$ But $h_0(z) = z \notin \mathbb{R}$.

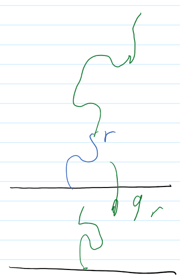
$T_z = \inf\{t : h_t(z) = 0\}$ - stopping time.
$h_t(z)$ is defined for $t \le T_z$.
For $t < T_z$, $z + B_t \in \Omega_{t/\kappa}$ (= component of ∞ of $\mathbb{H} \setminus \gamma[0,t]$)

Reminder: If $x \in \mathbb{R} \setminus \{0\}$, then
1) $\kappa \le 4$ ($\iff a \ge \frac{1}{2}$) $\Rightarrow T_x = \infty$ a.s.
2) $\kappa > 4$ ($\iff a < \frac{1}{2}$) $\Rightarrow T_x < \infty$ a.s.
3) $k \ge 8$ ($\iff a \le \frac{1}{4}$) $\Rightarrow_{a.s.} T_x < T_y$ $\forall x < y$
4) $4 < \kappa < 8$ ($\frac{1}{4} < a < \frac{1}{2}$), $x < y \Rightarrow P(T_x = T_y) > 0$.
5) $\kappa < 4 \Rightarrow$ A.s. $\forall x$ $\inf_t |h_t(x)| > 0$

Now to the proof:

Case $\kappa \le 4$.

$x \in \mathbb{R} \setminus \{0\} \Rightarrow \forall x\, T_x = \infty$ a.s. $\Rightarrow$ ($x \notin \overline{K_t}$ $\forall t < \infty$) $\Rightarrow$ $x \notin \gamma$ a.s. $\Rightarrow \gamma \cap \mathbb{R} \setminus \{0\} = \emptyset$
Use domain Markov property: a.s. $\forall$ rational r:
$P(g_r(\gamma(r,\infty)) \cap \mathbb{R} \setminus \{0\} \neq \emptyset) = 0 \iff P(\gamma(r,\infty) \cap \gamma(0,r) = \emptyset) = 1$.



If $t_1 < t_2$ and $\gamma(t_1) = \gamma(t_2)$, then $\exists s \in \mathbb{Q} : \gamma(s) \neq \gamma(t_1)$.
($\gamma[t_1, t_2] = \gamma(t_1) \Rightarrow K_{t_1} = K_{t_2}$, But hcap $K_{t_2} >$ hcap K_{t_1})
$t_1 < s < t_2$
Then $\gamma(0,s) \cap \gamma(s,\infty) \ni \{\gamma(t_1) = \gamma(t_2)\}$ - contradiction.
Thus γ is a simple curve.
To prove that $|\gamma(t)| \to \infty$ we need

Claim Let $0 < \kappa \le 4$. Then $P(\exists x \in \mathbb{R} \setminus \{0\}, t_n \uparrow \infty : \gamma(t_n) \to x) = 0$.
Proof. The condition can be restated as
$\mathrm{dist}(x, \gamma[0,\infty)) = 0$. By scaling and symmetry,
can assume $x \in [1, 1+\varepsilon]$ for some $\varepsilon > 0$.
Let us fix $\frac{1}{4} > \delta > 0$ and define $\sigma_\delta = \inf\{t : \mathrm{dist}(\gamma(t), [1, 1+\varepsilon]) \le \delta\}$.
(as usual, $\inf \emptyset = \infty$) - stopping time.

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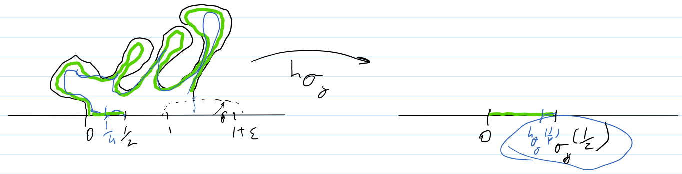
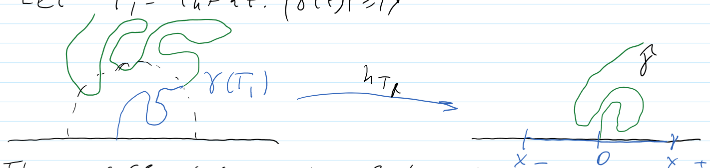


Let $R_\delta := [0, \frac{1}{2}] \cup$ right side of $\gamma[0, \sigma_\delta]$.

By Lavrentiev Theorem (after you map everything to a disk), we have that $h_{\sigma_\delta}(\frac{1}{2}) - 0 \le C\sqrt{\delta}$, so $\lim_{\delta \to 0} h_{\sigma_\delta}(\frac{1}{2}) = 0$. This finishes the proof when $\kappa < 4$,
$\inf_\delta h_{\sigma_\delta}(\frac{1}{2}) \ge \inf_t h_t(\frac{1}{2}) > 0$ – contradiction.

We need finer estimates of harmonic measure for $\kappa = 4$ – don't do here.

Return to $\gamma(t) \to \infty$. As $\operatorname{hcap} \gamma_t \to \infty$ ($t \to \infty$), γ_t is unbounded, i.e. $\exists\, t_k \uparrow \infty$, $|\gamma(t_k)| \to \infty$.
Let $T_1 = \inf\{t : |\gamma(t)| \ge 1\}$



There are two prime ends at 0 for $\mathbb{H} \setminus \gamma[0, T_1]$, let x_-, x_+ ($x_- < 0 < x_+$) be their images under h_{T_1}.
$\hat{\gamma}(t) := h_{T_1}(\gamma(t + T_1))$ – independent SLE_κ. By Claim, $\inf(\operatorname{dist}(\{x_-, x_+\}, \hat{\gamma})) > 0$. So, again by Lavrentiev,
a.s. $\exists\, r > 0 : |\gamma(T_1 + t)| \ge r \ \ \forall t \ge 0$.

Using scaling, $\forall n\, \exists r_n (\to 0) : \forall R : P(\exists s > t : |\gamma(s)| = R,\ |\gamma(t)| = r_n R) < 2^{-n}$.
Take $R_1 = 1$, $R_k = r_k^{-1} R_{k-1}$. $R_k \to \infty$.

<u>Case $4 < \kappa < 8$:</u>

$z \in \overline{\mathbb{H}}$ is called <u>swallowed</u> by $(K_t)_{t<\infty}$ if
$z \in \overline{K_{T_z}} \setminus \overline{\bigcup_{t<T_z} K_t}$.

<u>Observation:</u> $\overline{\mathbb{H}} \setminus \overline{\bigcup_{t<T_z} K_t}$ is relatively open in $\overline{\mathbb{H}}$,
z – swallowed $\Rightarrow z \notin \Omega_{T_z} \Rightarrow z$ belongs to a bounded component of $\overline{\mathbb{H}} \setminus \overline{\bigcup_{t<T_z} K_t} \Rightarrow \left(\exists\, \varepsilon = \varepsilon(z) : |w - z| < \varepsilon \Rightarrow w \text{ belongs to the same component of } \overline{\mathbb{H}} \setminus \overline{\bigcup_{t<T_z} K_t}\right) \Rightarrow$
w is swallowed. $T_z = T_w$. Can take $\varepsilon = \operatorname{dist}(z, \gamma(0, T_z)) = \operatorname{dist}(z, \gamma(0, \infty))$

<u>Know:</u> $\forall x \in \mathbb{R} \setminus \{0\}$, $P(T_x = T_1) > 0$.

Since, by Markov property and scaling, $P(T_x = T_1) \le P(T_{\sqrt[n]{x}} = T_1)^n$,

we get that $P(\exists x > 1 : T_x = T_1) = 1$.

Let $x_0 = \max\{x : T_x = T_1\} = \gamma(T_1)$.

Let $\varepsilon = \mathrm{dist}(1, \gamma(0, T_1)) > 0$, then $\forall w \in \overline{\mathbb{H}}$, $|w - 1| < \varepsilon \Rightarrow T_w = T_1$.

Thus γ is <u>not</u> space-filling.

If $0 \notin \partial K_{T_1}$, let $T' = T_1$. Otherwise, $T' := T_1 + \tilde{T}_{g_{T_1}(0)}$

($\tilde{T}_{g_t(0)}$ - swallowing time for $\gamma(T_1, \infty)$ - SLE_κ by Markov, $g_t(0)$ is swallowed with probability 1, by scaling)

In any case, $\mathrm{dist}(0, \partial K_{T'}) > 0$ a.s.

so $\forall \delta\ \exists \varepsilon, t : \overline{\mathbb{H}} \cap B(0, \varepsilon) \subset \overline{K_t}$ with probability $> 1 - \delta$.

By scaling, true for any $\varepsilon > 0$, which gives $\bigcup \overline{K_t} = \overline{\mathbb{H}}$.

<u>Case $\kappa \ge 8$.</u> A.s. $\forall y > x > 0 \Rightarrow T_x < T_y < \infty$

By symmetry, same is true for $0 > x > y$.

By the argument similar to the case $4 < \kappa < 8$ we see that $\forall z \in \mathbb{H}$, $T_z < \infty$.

Define, for $x \in \mathbb{R}$, $\Delta(x) = \mathrm{dist}(x + i, \gamma[0, \infty)) = \mathrm{dist}(x + i, \gamma[0, T_{x+i}])$

If we prove that $\Delta(x) = 0$ $\forall x$ a.s., then, by scaling a.s. $\forall z \in \mathbb{H}$, $\mathrm{dist}(z, \gamma([0, \infty))) = 0$, and γ is space-filling.

$h_t(z) := g_{t/\kappa}(z) - B_t$.

Let $h_t := X_t + i Y_t$, $X_t = \mathrm{Re}\, h_t$, $Y_t = \mathrm{Im}\, h_t$.

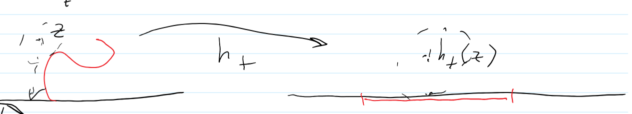
$t = \int_0^{\sigma(t)} \frac{ds}{X_s^2 + Y_s^2}$ - new time $\sigma(t)$.

Assume $\Delta(x) \ne 0$ for some x with positive probability,

By Köbe, $\underline{\mathrm{dist}(z, \gamma([0, t]))} \asymp \frac{\mathrm{Im}\, h_t(z)}{|h_t'(z)|} =: \exp(-D_t(z))$ — conformal radius of Ω_t at z.

so, if $D(x) := \lim_{t \to T_{x+i}} \ln \frac{|h_t'(x+i)|}{\mathrm{Im}\, h_t(x+i)}$, then $\Delta(x) \asymp \exp(-D(x))$.

$D_t(z) := \ln \frac{|h_t'(z)|}{\mathrm{Im}\, h_t(z)}$



<u>Need</u>: $D(x) = \infty$ a.s.

<u>Observe</u>:

$$\partial_t(\log|h_t'|) = \frac{2}{\kappa} \frac{Y_t^2 - X_t^2}{(X_t^2 + Y_t^2)^2}$$

$\partial_t(\log Y_t) = -\frac{2}{\kappa}\frac{1}{X_t^2+Y_t^2}$, so

$\partial_t D_t = \frac{4}{\kappa}\frac{Y_t^2}{(X_t^2+Y_t^2)^2}$. Thus

$D(x) = \frac{4}{\kappa}\int_0^\infty \frac{Y_s^2}{(X_s^2+Y_s^2)^2}\,ds$, $\quad D_t(x) = \frac{4}{\kappa}\int_0^t \frac{Y_s^2}{(X_s^2+Y_s^2)^2}\,ds$

Change time to $\sigma(t)$:

$\tilde{Y}_t := Y_{\sigma(t)}$; $\tilde{X}_t := X_{\sigma(t)}$; $\tilde{D}_t := D_{\sigma(t)}$; $\tilde{M}_t := \tilde{X}_t/\tilde{Y}_t$, $\tilde{C}_t := \log \tilde{M}_t$.

(But $D(x) = \tilde{D}(x)$!).

Then, by Itō:

$$d\tilde{M}_t = \frac{4}{\kappa}\tilde{M}_t\,dt + \sqrt{\tilde{M}_t^2+1}\,d\tilde{B}_t.$$

$$d\tilde{C}_t = \left(\frac{4}{\kappa} - \frac{1}{2} - \frac{1}{2}e^{-2\tilde{C}_t}\right)dt + \sqrt{1+e^{-2\tilde{C}_t}}\,d\tilde{B}_t.$$

$$d\tilde{D}_t = \frac{4}{\kappa}\frac{\tilde{Y}_t^2\,dt}{\tilde{X}_t^2+\tilde{Y}_t^2} = \frac{4}{\kappa}\frac{1\,dt}{1+e^{2\tilde{C}_t}}$$

From this we get, since $\tilde{D}_0(x)=0$,

$$\tilde{D}(x) = \frac{4}{\kappa}\int_0^\infty \frac{ds}{e^{2\tilde{C}_s}+1}.$$

Note that $\frac{4}{\kappa} - \frac{1}{2} - \frac{1}{2}e^{-2\tilde{C}_t} < 0$ ($\kappa \geq 8$!)

so $\forall\, T>0 \;\; \exists\, t>T:\; \tilde{C}_s \leq 0 \;\; \forall\, s \in [t, t+1]$

so $\tilde{D}(x) = \infty$ a.s. $\blacksquare$

Remark. Same method gives the distribution of $D(x)$ when $\kappa < 8$:

$$E(e^{i\theta D(x)}) = \frac{\Gamma\left(\frac{2}{\kappa} + \sqrt{(\frac{2}{\kappa}-\frac{1}{2})^2 - i\frac{2\theta}{\kappa}}\right)\Gamma\left(\frac{2}{\kappa} - \sqrt{(\frac{2}{\kappa}-\frac{1}{2})^2 - i\frac{2\theta}{\kappa}}\right)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{4}{\kappa}-\frac{1}{2})}\; F(\alpha_+, \alpha_-, \tfrac{1}{2}, x).$$

F - hypergeometric function,

$$\alpha_\pm = \frac{1}{2} - \frac{2}{\kappa} \pm \sqrt{\left(\frac{2}{\kappa}-\frac{1}{2}\right)^2 - i\frac{2\theta}{\kappa}}.$$

We'll need for the dimension!

Proof, If $\kappa < 8$, define new time $\hat{\sigma}_t$ by $\partial_t \hat{\sigma}(t) = \frac{1}{\tilde{M}_t^2+1} = \frac{\tilde{Y}_t^2}{\tilde{X}_t^2+\tilde{Y}_t^2} = (\sin \arg \tilde{h}_t)^2$

Then, in this coordinates:

$\hat{M}_t := \tilde{M}_{\hat{\sigma}(t)}$, $\quad$ B.M. in $\hat{\sigma}$.

$$d\hat{M}_t = \frac{4}{\kappa}\frac{\hat{M}_t}{1+\hat{M}_t^2}\,dt + d\hat{B}_t = \frac{4}{\kappa}\frac{\hat{X}_t\hat{Y}_t}{\hat{X}_t^2+\hat{Y}_t^2}\,dt + d\hat{B}_t.$$

$$\frac{2}{\kappa}\sin(2\arg \hat{h}_t)\,dt + d\hat{B}_t. \qquad \hat{X}_t$$

$$\frac{2}{\kappa}\sin\left(2\arg\widehat{h_t}\right)dt + d\widehat{B}_t.$$

$$\frac{\widehat{X}_t}{\sqrt{\widehat{X}_t^2+\widehat{Y}_t^2}} = \cos\arg\widehat{h_t}$$

$$\frac{\widehat{Y}_t}{\sqrt{\widehat{X}_t^2+\widehat{Y}_t^2}} = \sin\arg\widehat{h_t}$$

and $D(x) = \frac{4}{\kappa}\int_0^\infty \frac{dt}{(1+\widehat{M}_t^2)^2}dt$

Richard Feynman (1918-1988)

Mark Kac (1914-1984)

We will use the following version of

Feynman-Kac formula:

Let 1) Y_t satisfies the SDE:

$$dY_t = F(Y_t)dt + dB_t, \quad Y_0 = y$$

2) $V: \mathbb{R} \to \mathbb{R}_+$ - continuous.

3) $Y_t \to \infty$ $(t\to\infty)$ a.s

4) $\int_0^\infty V(Y_t)\,dt < \infty$ a.s

5) $\psi: \mathbb{R} \to \mathbb{C}$, bounded, $\in C^2$, satisfies

$$\frac{1}{2}\psi'' + F\psi' + iV\psi = 0, \quad \lim_{y\to\infty}\psi(y) = 1.$$

Then $\psi(y) = E\left(\exp\left(i\int_0^\infty V(y_t)\,dt\right)\right)$

Proof. (of FK) By Itô:

$M_t := \psi(Y_t)\exp\left(i\int_0^t V(Y_s)\,ds\right)$ - bounded martingale

(drift term disappears because of 5))

By martingale convergence theorem,

$$\psi(y) = M_0 = E\left(\lim_{t\to\infty} M_t\right) = E\left(i\int_0^\infty V(y_t)\,dt\right)$$

Apply FK with ψ being the solution

to $\frac{1}{2}\psi''(y) + \frac{4y}{\kappa(1+y^2)}\psi'(y) + \frac{2b}{\kappa(1+y^2)^2}\psi(y) = 0$

which is bounded, $\lim_{y\to\infty}\psi(y) = 1$.

with $y_t = C\widehat{M}_t$ (with $C\frac{X_0}{Y_0} = b$)

and $\underbrace{V = \frac{b}{\kappa(1+x^2)^2}}$.

$x(1+x^2)$